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THE ASYMPTOTIC STABILITY OF THE EQUILIBRIUM OF NON-AUTONOMOUS SYSTEMS†

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A method of investigating the stability of non-linear systems acted upon by unsteady perturbations is proposed, based on the use of Lyapunov's second method. The sufficient conditions for asymptotic stability of the solutions of non-autonomous systems in critical cases are obtained.

1. Consider the following system of differential equations

$$\dot{x}_s = f_s(\mathbf{X}), \quad s = 1, \dots, n \tag{1.1}$$

In addition to system (1.1) we will also consider the perturbed system

$$\dot{x}_s = f_s(\mathbf{X}) + \sum_{j=1}^{k} b_{sj}(t) h_{sj}(\mathbf{X}), \quad s = 1, \dots, n \tag{1.2}$$

The functions $f_s(\mathbf{X})$ and $h_{sj}(\mathbf{X})$ are defined and continuously differentiable for all $\mathbf{X} \in R^n$, and the functions $b_{sj}(t)$ are continuous and bounded for $t \geqslant 0$.

We will assume that systems (1.1) and (1.2) have a zero solution $\mathbf{X} = 0$, and that the zero solution of (1.1) is asymptotically stable. We will investigate the conditions for which the zero solution of the perturbed system is also asymptotically stable.

We know [1–3], that if $f_s(\mathbf{X})$ are homogeneous functions, perturbations will not disturb the asymptotic stability of the zero solution when their order is higher than the order of the right-hand sides of system (1.1).

It was assumed in [4] that $f_s(\mathbf{X})$ and $h_{sj}(\mathbf{X})$ are homogeneous functions of order μ and σ, respectively, while the integrals

$$I_{sj}(t) = \int_0^t b_{sj}(\tau) d\tau \tag{1.3}$$

are bounded when $t \in [0, +\infty]$. It was shown that for this type of perturbations asymptotic stability can also be preserved when $\sigma \leqslant \mu$.

In this paper we consider the case when $f_s(\mathbf{X})$ and $h_{sj}(\mathbf{X})$ are generalized homogeneous functions, while the integrals (1.3), generally speaking, are not bounded when $t \geqslant 0$.

2. Suppose the function $g(\mathbf{X})$ is specified and continuous for all $\mathbf{X} \in R^n$.

Definition 5. The function $g(\mathbf{X})$ is called a generalized homogeneous function of the class $(m_1, \dots, m_n)$ of order m if

$$g\left(c^{m_1} x_1, \dots, c^{m_n} x_n\right) = c^m g\left(x_1, \dots, x_n\right), \quad \forall c \in (-\infty; +\infty)$$

where $m, m_1, \dots m_n$ are positive rational numbers with odd denominators.

We will assume that $f_s(\mathbf{X})$ and $h_{sj}(\mathbf{X})$ are generalized homogeneous functions of the class $(m_1, \dots, m_n)$ of order $m_s + \mu$ and $m_s + \sigma$ respectively, where μ and σ are positive rational numbers with odd denominators.

We know [5] that it follows from the asymptotic stability of the zero solution of system (1.1) that positive-definite generalized homogeneous functions $V(\mathbf{X})$ and $W(\mathbf{X})$ of the class $(m_1, \dots, m_n)$ of order

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m and $m+\mu$ exist, for which the following equation holds

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} f_s(\mathbf{X}) = -W(\mathbf{X})$$

where the function $V(\mathbf{X})$ is continuously differentiable. Using these functions we can establish that when $\sigma > \mu$ the zero solution of system (1.2) is asymptotically stable.

Suppose the function $V(\mathbf{X})$ is twice continuously differentiable. For this to be so it is sufficient for the function $f_s(\mathbf{X})$ to be twice continuously differentiable [5].

We will also assume that the integrals (1.3), generally speaking, are not bounded when $t \geqslant 0$. However, a number α, $0 \leqslant \alpha \leqslant 1$ exists such that

$$\lim_{t\to+\infty} \frac{1}{t^{\alpha}} I_{sj}(t) = 0$$

Theorem. When the inequality

$$2\sigma \geqslant (\alpha + 1)\mu \tag{2.1}$$

is satisfied the zero solution of system (1.2) is asymptotically stable.

Proof. Consider the function

$$V_1 = V(\mathbf{X}) - \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{k} I_{sj}(t) h_{sj}(\mathbf{X})$$

By virtue of system (1.2) its derivative has the form

$$\frac{dV_1}{dt} = -W(\mathbf{X}) - \sum_{s=1}^{n} \sum_{j=1}^{k} I_{sj}(t) \sum_{r=1}^{n} \frac{\partial}{\partial x_r}\left(\frac{\partial V}{\partial x_s} h_{sj}\right)\left(f_r(\mathbf{X}) + \sum_{i=1}^{k} b_{ri}(t) h_{ri}(\mathbf{X})\right)$$

Suppose

$$Z(\mathbf{X}) = \sum_{s=1}^{n} |x_s|^{1/m_s}, \quad \varphi(t) = \sum_{s=1}^{n} \sum_{j=1}^{k} |I_{sj}(t)|$$

For all $\mathbf{X} \in R^n$ and $t \geqslant 0$ we have the inequalities

$$a_1 Z^m - a_3 \varphi(t) Z^{m+\sigma} \leqslant V_1(t, \mathbf{X}) \leqslant a_2 Z^m + a_3 \varphi(t) Z^{m+\sigma}$$

$$\frac{dV_1}{dt} \leqslant -c_1 Z^{m+\mu} + \varphi(t)\left(c_2 Z^{m+\mu+\sigma} + c_3 Z^{m+2\sigma}\right)$$

where $a_1, a_2, a_3, c_1, c_2, c_3$ are positive constants [5].

We will choose the numbers δ, γ, A and T so that the following conditions are satisfied

$$3ma_2 > \mu c_1 \gamma, \quad \mu c_1 A a_1^{\mu/m} > m(3a_2)^{1+\mu/m}$$

$$2a_3 \delta A^{\sigma/\mu} < a_1, \quad 4c_2 \delta A^{\sigma/\mu} < c_1, \quad 4c_3 \delta A^{2\sigma/\mu - 1} < c_1$$

$$\varphi(t) < \sigma t^{\alpha} \quad \text{when} \quad t \geqslant T$$

Consider the solution $\mathbf{X}(t) = \mathbf{X}(t, \mathbf{X}_0, t_0)$ of system (1.2), the initial data of which satisfy the conditions $Z^{\mu}(\mathbf{X}_0) < \gamma/t_0$, $t_0 \geqslant T$.

For all $t \geqslant t_0$ the following inequality holds

$$Z^{\mu}(\mathbf{X}(t)) < A/t \tag{2.2}$$

In fact, if an instant of time $t_1 > t_0$ exists such that $Z^{\mu}(\mathbf{X}(t_1)) = A/t_1$, and condition (2.2) is satisfied for $t \in [t_0, t_1)$, then for $t \in [t_0, t_1]$ we have

$$\frac{a_1}{2} Z^m(\mathbf{X}(t)) \leqslant V_1(t,\mathbf{X}(t)) \leqslant \frac{3a_2}{2} Z^m(\mathbf{X}(t))$$

$$\frac{dV_1(t,\mathbf{X}(t))}{dt} \leqslant -\frac{c_1}{2} Z^{m+\mu}(\mathbf{X}(t))$$

Using the method of estimates [5] we arrive at the inequality

$$t_1\left(\frac{\mu c_1}{3ma_2} - \frac{1}{A}\left(\frac{3a_2}{a_1}\right)^{\mu/m}\right) \leqslant t_0\left(\frac{\mu c_1}{3ma_2} - \frac{1}{\gamma}\right)$$

It follows from the conditions for choosing the numbers γ, A and T that the left-hand side of this inequality is positive while the right-hand side is negative. We obtain a contradiction.

Using the proved property of the solutions of system (1.2) and also their continuous dependence on the initial data, we obtain that the zero solution of system (1.2) is asymptotically stable. This proves the theorem.

Corollary 1. If the integrals (1.3) are bounded for $t \geqslant 0$, then when the inequality

$$2\sigma > \mu$$

is satisfied the zero solution of system (1.2) is asymptotically stable.

Corollary 2. Suppose $f_s(\mathbf{X})$ and $h_{sj}(\mathbf{X})$ are homogeneous functions of order $\mu + 1$ and $\sigma + 1$, respectively. Then, when inequality (2.1) is satisfied the zero solution of system (1.2) is asymptotically stable.

Hence, we obtain that for this form of perturbations the asymptotic stability of the zero solution of system (1.2) can also be preserved when $\sigma \leqslant \mu$.

3. We will consider some examples of the use of the above theorem.

Example 1. Suppose the motion of a mechanical system is described by the equations

$$\ddot{\mathbf{q}} = -\frac{\partial P}{\partial \mathbf{q}} + \frac{\partial W}{\partial \dot{\mathbf{q}}} \tag{3.1}$$

Here $\mathbf{q}$ is an n-dimensional vector of the generalized coordinates, the function $P(\mathbf{q})$ corresponds to the potential energy of the system and is a twice continuously differentiable positive-definite homogeneous function of order λ, $\lambda > 2$, and $W(\dot{\mathbf{q}})$ is a twice continuously differentiable negative-definite homogeneous function of order μ, $\mu > 2$. Hence, system (3.1) is dissipative and the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = 0$ is asymptotically stable [6].

We will assume that $\mu < 3$ while $\lambda = 2/(3 - \mu)$. Then, the system considered is a generalized homogeneous system, and the generalized homogeneous Lyapunov function can be chosen in the form

$$V = \left(\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}} + P(\mathbf{q})\right)^l + c\sum_{s=1}^{n} \dot{q}_s\left(\frac{\partial P}{\partial q_s}\right)^\beta, \quad \beta = \frac{\lambda(2l-1)}{2\lambda - 2}$$

where l is a natural number and c is a positive constant.

For sufficiently small c the function V is positive definite, and its derivative, calculated by virtue of system (3.1), is a negative-definite function.

We will also assume that the functions $(\partial P/\partial q_s)^\beta$ are twice continuously differentiable, at least for fairly high values of β.

We will now consider the perturbed system

$$\ddot{\mathbf{q}} = -\frac{\partial P}{\partial \mathbf{q}} + \frac{\partial W}{\partial \dot{\mathbf{q}}} + \mathbf{B}_1(t)\mathbf{H}(\dot{\mathbf{q}}) + \mathbf{B}_2(t)\mathbf{R}(\mathbf{q}) \tag{3.2}$$

where the elements of the vectors $\mathbf{H}(\dot{\mathbf{q}})$ and $\mathbf{R}(\mathbf{q})$ of dimensionality k and m are continuously differentiable homogeneous functions of order ν and σ respectively, $\nu < 1, \sigma > 1$, and $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ are $n \times k$ and $n \times m$ matrices, continuous and bounded for $t \geqslant 0$, and a number α, $0 \leqslant \alpha \leqslant 1$ exists such that

$$\lim_{t\to+\infty} \frac{1}{t^\alpha}\int_0^t \mathbf{B}_j(\tau)d\tau = \mathbf{0}, \quad j = 1,2$$

Using the above theorem, we obtain the sufficient conditions for asymptotic stability of the position of equilibrium $\mathbf{q} = \dot{\mathbf{q}} = 0$ of system (3.2)

$$2\nu \geqslant \alpha(\mu-2)+\mu, \quad 2\sigma \geqslant (\alpha(\mu-2)+\mu)/(3-\mu)$$

Example 2. Consider the Lienard vector equation

$$\ddot{\mathbf{X}} + \frac{\partial \mathbf{F}}{\partial \mathbf{X}}\dot{\mathbf{X}} + \frac{\partial G}{\partial \mathbf{X}} = \mathbf{0} \tag{3.3}$$

Here $\mathbf{X}$ is an n-dimensional vector of the unknown functions, and the components of the vector $\mathbf{F}(\mathbf{X})$ and the scalar function $\mathbf{G}(\mathbf{X})$ are twice continuously differentiable homogeneous functions of order μ and 2μ respectively, $\mu > 2$.

Making the replacement $\mathbf{Y} = \dot{\mathbf{X}} + \mathbf{F}(\mathbf{X})$ in Eq. (3.3), we obtain the generalized homogeneous system

$$\dot{\mathbf{X}} = \mathbf{Y} - \mathbf{F}(\mathbf{X}), \quad \dot{\mathbf{Y}} = -\frac{\partial G}{\partial \mathbf{X}}$$

We will assume that the functions $\mathbf{G}(\mathbf{X})$ and $\mathbf{F}^T\,\partial G/\partial \mathbf{X}$ are positive definite. Then, the zero solution of this system is asymptotically stable, while the twice continuously differentiable generalized homogeneous Lyapunov function can be expressed in the form

$$V = \left(\frac{1}{2}\mathbf{Y}^T\mathbf{Y} + G(\mathbf{X})\right)^2 + c\sum_{s=1}^{n} x_s y_s^{4-1/\mu}$$

where c is a negative constant.

If the number c is sufficiently small in absolute value, the function constructed satisfies the conditions of Lyapunov's theorem on asymptotic stability.

In addition to Eq. (3.3) we will consider the perturbed equation

$$\ddot{\mathbf{X}} + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{X}} + \mathbf{B}_1(t)\mathbf{H}(\mathbf{X})\right)\dot{\mathbf{X}} + \frac{\partial G}{\partial \mathbf{X}} + \mathbf{B}_2(t)\mathbf{R}(\mathbf{X}) = \mathbf{0} \tag{3.4}$$

We will assume that the elements of the $k \times n$ matrix $\mathbf{H}(\mathbf{X})$ and the components of the m-dimensional vector $\mathbf{R}(\mathbf{X})$, and also the $n \times k$ and $n \times m$ matrices $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ possess the properties indicated in Example 1.

The conditions for asymptotic stability of the zero solution of Eq. (3.4) then have the form

$$2\nu \geqslant (\alpha+1)(\mu-1), \quad 2\sigma \geqslant 2\mu + (\alpha+1)(\mu-1) \tag{3.5}$$

Notes. 1. In some cases the results obtained in this paper can be extended by using a somewhat modified method for constructing the Lyapunov functions (see [7]).

For examples, applying this method to Eq. (3.4) we obtain new conditions for asymptotic stability of the zero solution

$$2\nu \geqslant (\alpha+1)(\mu-1), \quad \sigma \geqslant \mu + \alpha(\mu-1), \quad \nu+\sigma \geqslant \mu + (\alpha+1)(\mu-1)$$

which refine conditions (3.5).

2. For some types of non-linear systems the proposed method of investigating the stability of the equilibrium position can also be used when the right-hand sides of the unperturbed equations are not generalized homogeneous functions.

Example 3. Consider a solid rotating around a fixed point O, situated at its centre of inertia. We will assume that the axes $Oxyz$, which are the principal central axes of this body, are connected with the body. The equations of rotational motion of the body acted upon by a controlling moment $\mathbf{M}$ have the form

$$\Theta\dot{\omega} + \omega \times \Theta\omega = \mathbf{M} \tag{3.6}$$

where ω is the angular velocity vector and Θ is the inertia tensor of the body [8].

Suppose two unit vectors $\mathbf{r}$ and $\mathbf{s}$ are specified; the vector $\mathbf{s}$ will be assumed to be fixed in absolute space while the vector $\mathbf{r}$ is fixed in the solid. The vector $\mathbf{s}$ then rotates with respect to the system $Oxyz$ with angular velocity $-\omega$. Consequently

$$\dot{\mathbf{s}} = -\omega \times \mathbf{s} \tag{3.7}$$

We will construct the controlling moment $\mathbf{M}$ for which the system of equations (3.6) and (3.7) has an asymptotically stable position of equilibrium $\mathbf{s} = \mathbf{r}$, $\omega = \mathbf{0}$.

We know [8], that the moment **M** can be chosen in the form

$$\mathbf{M}=\frac{\partial W}{\partial \omega}-\frac{dG}{d\varphi}\mathbf{s}\times\mathbf{r}$$

where $W(\omega)$ is a negative-definite functions of the components of the vector ω and $G(\varphi)$ is a positive-definite function of the quantity $\varphi = \|\mathbf{s}-\mathbf{r}\|^2/2$.

We will assume that $W(\omega)$ is a twice continuously differentiable homogeneous function of order μ, $\mu > 2$; $G(\varphi) = a\varphi^{\lambda+1}$, where $a > 0$, $\lambda \geq 1$.

Suppose $\mathbf{X} = \mathbf{s}\times\mathbf{r}$, $\mathbf{Y} = (x_1^\beta, x_2^\beta, x_3^\beta)^T$, where β is a rational number with odd numerator and denominator, $\beta \geq 1$.

Consider the function

$$V=\frac{1}{2}\omega^T\Theta\omega+G(\varphi)+c\omega^T\Theta\mathbf{Y} \tag{3.8}$$

where c is a positive constant.

If $\beta \geq \max\{1 + (\mu-2)(\lambda + 1), (2\lambda + 1)/(\mu-1)\}$, and the number c is sufficiently small, the function V is positive definite, and its derivative, calculated by virtue of Eqs (3.6) and (3.7), is a negative-definite function.

Using this function it can be shown that to solve Eqs (3.6) and (3.7), beginning at $t = 0$ in a fairly small neighbourhood of the equilibrium position $\mathbf{s} = \mathbf{r}$, $\omega = \mathbf{0}$, for all $t \geq 0$, we have the inequalities

$$\|\omega(t)\|\leq A(t+1)^{-(\lambda+1)/\delta},\quad \|\mathbf{s}(t)-\mathbf{r}\|\leq A(t+1)^{-1/\delta}$$

Here $A > 0$, $\delta = \max\{(\mu-2)(\lambda + 1), \beta-1\}$.

Suppose that the body considered is acted upon, in addition to the controlling moment, by the moment of external perturbing forces $\mathbf{M}_1$, having the form

$$\mathbf{M}_1 = \mathbf{B}(t)\mathbf{H}(\omega)$$

where the components of the k-dimensional vector $\mathbf{H}(\omega)$ are continuously differentiable homogeneous functions of order ν, $\nu > 1$ and $\mathbf{B}(t)$ is a $3\times k$ matrix, continuous and bounded for $t \geq 0$.

Differentiating the Lyapunov function (3.8) by virtue of the perturbed system we obtain the sufficient condition for asymptotic stability of the equilibrium position investigated

$$\nu>\mu-1 \tag{3.9}$$

We will further assume that a number α, $0 \leq \alpha \leq 1$ exists such that

$$\lim_{t\to+\infty}\frac{1}{t^\alpha}\int_0^t \mathbf{B}(\tau)d\tau=\mathbf{0}$$

Condition (3.9) can then be refined by using the method proposed in this paper for investigating the stability of the solutions of non-autonomous systems.

We will choose the Lyapunov function for the perturbed system in the form

$$V_1=V-(\omega+\mathbf{Y})^T\int_0^t \mathbf{B}(\tau)d\tau\mathbf{H}(\omega)$$

Using this function we obtain that, when the following inequalities are satisfied

$$\nu\geq\max\left\{\frac{\mu}{2}+\frac{\alpha(\mu-2)}{2},\mu-2+\frac{1}{\lambda+1}+\alpha(\mu-2),1+\xi,\frac{\mu}{2}+\frac{\xi}{2}\right\},\quad \left(\xi=\frac{\alpha(2\lambda+2-\mu)}{(\lambda+1)(\mu-1)}\right) \tag{3.10}$$

the equilibrium position $\mathbf{s} = \mathbf{r}$, $\omega = \mathbf{0}$ of the perturbed system is asymptotically stable.

Note. For systems with generalized homogeneous right-hand sides, condition (2.1) for all $\alpha \in [0, 1]$ refines the well-known condition for the asymptotic stability of the zero solution $\sigma > \mu$. In the example considered, inequality (3.10) extends the set of values of the parameter ν, defined by condition (3.9), only in the case when the following inequalities are satisfied

$$\alpha(\mu-2)(\lambda+1)\leq\lambda,\quad \alpha(2\lambda+2-\mu)\leq(\mu-1)(\mu-2)(\lambda+1)$$

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